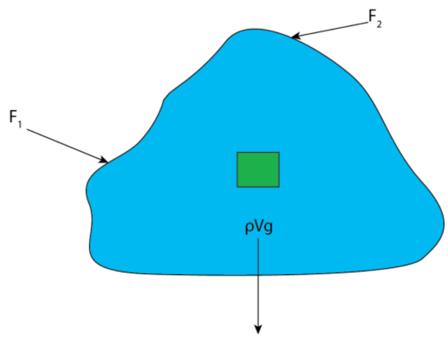
Strain

1.0 Strain

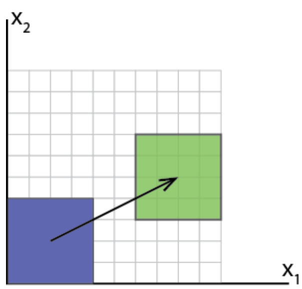
Consider a square of material in a 2-D substance that is being acted on by surface and body forces.



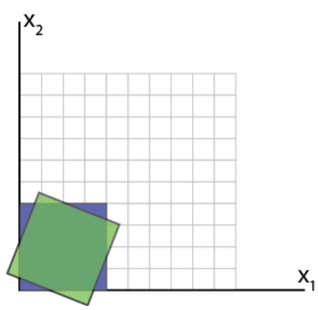
Stress on the square can have three effects:

rigid body motions

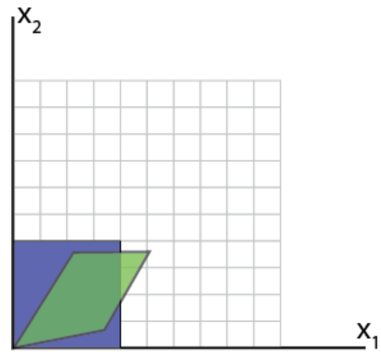
1) Translation



2) Rotation

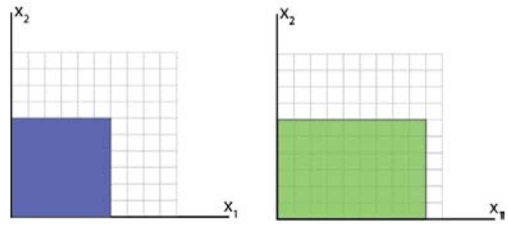


3) Deformation (Strain)

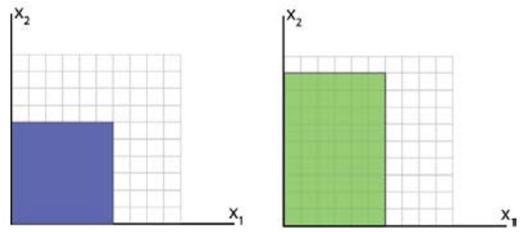


If you think about it, there are 3 ways the square can be strained:

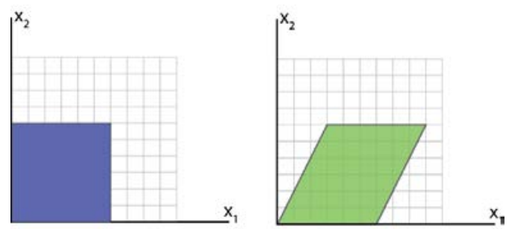
1) Change its size in the x_1 direction.



2) Change its size in the x_2 direction



3) Shear it.

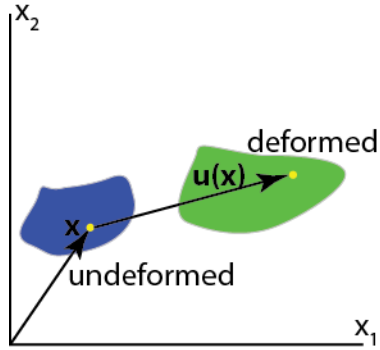


Like with stress, we have normal strains (1 and 2 above), and shear strains (3), and can represent strain as a tensor. Let's examine how to quantify strain, which is formally defined as the relative change in particle distances in an infinitesimal body.

First note that we will talk about things in terms of a 1, 2, 3 coordinate system (i.e. v_1 is the x-component of vector **v**, v_2 is the y-component of vector **v**, and v_3 is the z-component of vector **v**). Vector **x** is thus an arbitrary vector in 3 dimensions, and has nothing to do with a particular axis. If you find it frustrating, this is done because it makes it easier to use

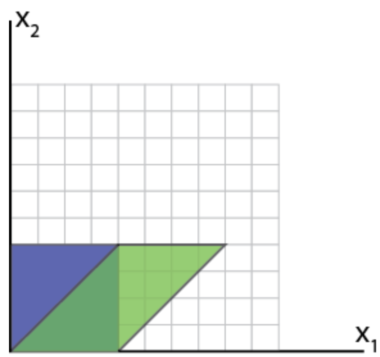
Einstein Summation Notation. Also note that you will see this sort of notation used in other books, so it is useful to get a handle on this.

Imagine a solid body where displacements **u**(**x**) have been induced by stress. For any point, **x** = (x_1, x_2), **u(x)** will describe how far that point is moved by the deformation.



So, imagine our point **x** is at **x** = (3,3). If deformation occurs, and **u(x)** = (1,1), then the new location of the point is at **x'** = **x** + **u(x)** = (4,4). Basically, this **u(x)** moves everything up and to the right by a distance of sqrt(2). In this case, a square before the deformation would remain a square after the deformation, and have the same volume.

Now, imagine a **u(x)** that was slightly more complicated, say **u(x)** = (x_2,0). In this case, a point at **x** = (0,0) would remain at 0,0. But for a point at **x** = (0,1), **u(x)** would be (1,0). Hence, the new location of the point would be **x'** = (1, 1). So, a square would be sheared, but keep the same volume, as you can see below.

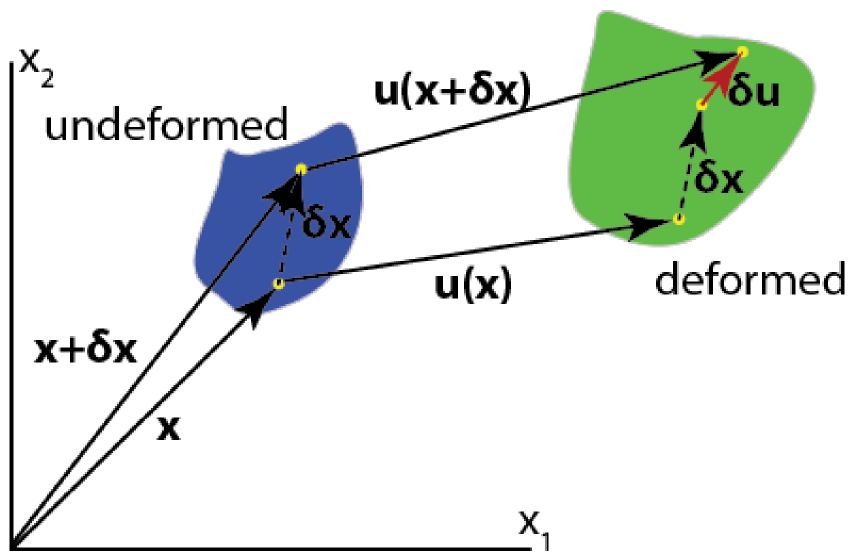


We can imagine more complicated versions of **u(x)**. For instance, **u(x)** = (x_1^2, x_2^2). If square 1 has vertices at (0,0), (0,1), (1,1), and (1,0), it would be transformed to (0,0), (0,2),(2,2), and (2,0). Square 2 at (2,2),(2,3),(3,3), and (3,2) would be transformed to (6,6),(6,12),(12,12), and (12,6). Note that square 1 has its volume change by a factor of 4, but square 2 has a volume change of a factor of 36. So the position of the square affects the volume change. Of course deformation in the x_1 direction could also be a function of position in the x_2 direction. So **u(x)** could be something like **u(x)** = ($x_2\, x_1^2$, x_1+x_2).

Our goal is to come up with a tensor that defines the deformation of an infinitesimal cube located at a point **x**. It should be able to represent shear (as in the figure above) and volume change. Since the effect of **u(x)** on a vertex of a cube is dependent upon position, we will have to consider how the effect of **u** *changes* with position. As soon as you hear the words, *function*, *change*, and *position*, you should be thinking: derivative, since that is what a derivative measures: the change in a function as position (or another variable) changes.

Since we are interested in strain in an elastic body, we can assume **u(x)** varies smoothly from point to point. Mathematically, this means that if **u(x)** represents the deformation at point **x**, we can come up with a function **u(x) + δu(x)** that is the deformation at nearby point **x + δx.** This is represented graphically by the figure below, from the *Stein and Wysession* seismology book.

We are getting somewhere, believe it or not, because now we have a measure, **δu**, that describes the displacement of any two nearby points due to deformation. If we can get a handle on how nearby points (like the vertices of a cube) move, we have a way to fully represent deformation. And since strain is defined as the relative movement of nearby particles in a medium, this will give us strain.



What we are saying here is that the initially the two particles are separated by vector $\delta\vec{x}$; after deformation, the new separation is $\delta\vec{u}$. We assume that there is a function that describes how $\delta\vec{u}$ varies with $\vec{x}$. By the Taylor series approximation[1] (i.e. this only works for small values of $\delta\vec{x}$),

$$u_i(\vec{x}+\delta\vec{x}) \approx \underbrace{u_i(\vec{x})}_{displacement} + \underbrace{\vec{\nabla}\vec{u}}_{rotation+strain} + \underbrace{\text{higher order terms}}_{\text{ignored since } \delta\vec{x} \text{ is small}} = u_i(\vec{x}) + \frac{\partial u_i(x)}{\partial x_j}\delta x_j = u_i(\vec{x}) + \delta u_i$$

So,

[1] We only need the first term of the Taylor Series because we are considering small displacements. There is a formalism for dealing with large displacements, such as might be found in ductile flow, but we won't discuss these here and will consider only elastic deformation.

$$\delta u_i(\vec{x}) = \frac{\partial u_i(x)}{\partial x_j}\delta x_j$$

Here we are using ESN; since *j* is repeated in the left hand side of the above equation, we assume summation over *j*. In long-hand, the equation is

$$\delta u_i(\vec{x}) = \frac{\partial u_i(x)}{\partial x_1}\delta x_1 + \frac{\partial u_i(x)}{\partial x_2}\delta x_2 + \frac{\partial u_i(x)}{\partial x_3}\delta x_3$$

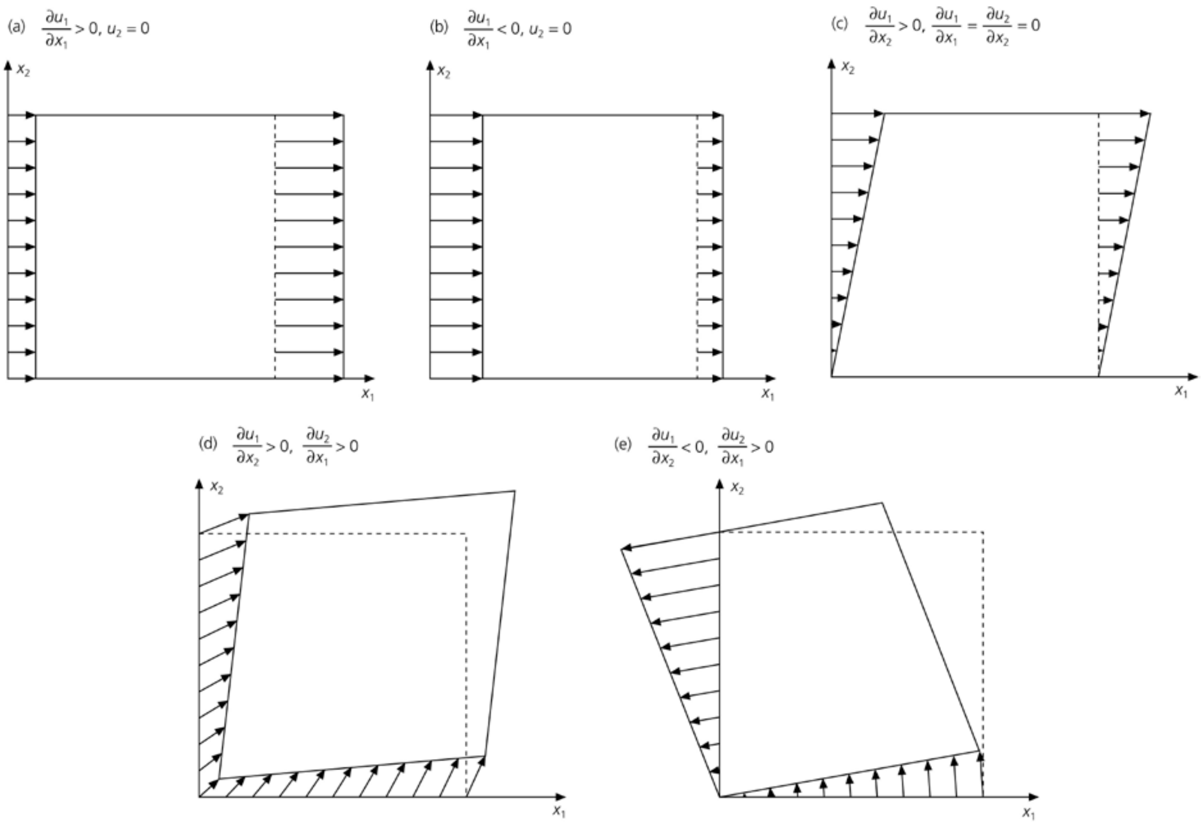
There are three of these equations, for *i* = 1, 2, and 3. So, we can write it out as

$$\delta u_i(\vec{x}) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}.$$

I will digress for a bit here, and you can ignore this box if desired. In the Taylor Series above, I used the vector gradient function: $\vec{\nabla}\vec{u}$. This is a little different from the standard scalar gradient one finds in a vector calc class. The scalar gradient operates on a scalar field, such as a series of elevations. It gives the direction of maximum change, so in the case of elevations, the scalar gradient tells you the direction downhill. Since it is giving a direction, that means it is taking a bunch of scalar values (a field), and returning a vector. More formally, if *f* defines a function ($f = f(\vec{x})$), then $\nabla f = \frac{\partial f}{\partial x_1}\hat{x}_1 + \frac{\partial f}{\partial x_2}\hat{x}_2 + \frac{\partial f}{\partial x_3}\hat{x}_3$. In the case of a vector function, such as $\vec{u}(\vec{x})$ that maps a vector to another vector, the gradient of this is a 2nd order tensor.
$\vec{\nabla}\vec{u}(\vec{x}) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \end{bmatrix}$. It turns out the gradient of the displacement function defines the strain and rotation of the material in question. I mention this because one will occasionally see this used in the derivation of the strain tensor, and if you are thinking this is a scalar gradient it will be confusing.

Before we go any further in using derivatives of the deformation function to define strain and rotation, which is our goal, let's look at some examples to get an intuitive feel for how the partial derivatives of **u** with respect to **x** affect the shape of a square.

Figure 2.3-12: Some possible strains for a two-dimensional element.



Note that the mixed derivatives tend to shear the square, where as the non-mixed derivatives stretch it. Note that (c) and (e) in the figure above have non-zero rotation terms. (c) shows simple shear. None of these figures show pure shear, which is a case in which the volume change is 0, there is no rotation, and the body shrinks in one dimension and increases in the two other dimensions.

Now, back to the three equations $\delta u_i(\vec{x}) = \frac{\partial u_i(x)}{\partial x_1}\delta x_1 + \frac{\partial u_i(x)}{\partial x_2}\delta x_2 + \frac{\partial u_i(x)}{\partial x_3}\delta x_3$. We can separate each equation into two parts:

$$\delta u_i = \frac{1}{2}\left[\frac{\partial u_i(x)}{\partial x_1}\delta x_1 + \frac{\partial u_i(x)}{\partial x_2}\delta x_2 + \frac{\partial u_i(x)}{\partial x_3}\delta x_3\right] + \frac{1}{2}\left[\frac{\partial u_i(x)}{\partial x_1}\delta x_1 + \frac{\partial u_i(x)}{\partial x_2}\delta x_2 + \frac{\partial u_i(x)}{\partial x_3}\delta x_3\right]$$

In ESN, this gives us

$$\delta u_i(\vec{x}) = \frac{1}{2}\frac{\partial u_i(x)}{\partial x_j}\delta x_j + \frac{1}{2}\frac{\partial u_i(x)}{\partial x_j}\delta x_j$$

Now add $\frac{\partial u_j}{\partial x_i}$ to the first part, and subtract it from the second part

$$\delta u_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j$$

.

We'll do this in semi-long hand, by writing out the summation for *j*:

$$\delta u_i = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_1}\delta x_1 + \frac{\partial u_1}{\partial x_i}\delta x_1 + \frac{\partial u_i}{\partial x_2}\delta x_2 + \frac{\partial u_2}{\partial x_i}\delta x_2 + \frac{\partial u_i}{\partial x_3}\delta x_3 + \frac{\partial u_3}{\partial x_i}\delta x_3\right] +$$
$$\frac{1}{2}\left[\frac{\partial u_i}{\partial x_1}\delta x_1 - \frac{\partial u_1}{\partial x_i}\delta x_1 + \frac{\partial u_i}{\partial x_2}\delta x_2 - \frac{\partial u_2}{\partial x_i}\delta x_2 + \frac{\partial u_i}{\partial x_3}\delta x_3 - \frac{\partial u_3}{\partial x_i}\delta x_3\right]$$

In ESN, where summation over the repeated *j* indices is assumed, the equation becomes much simpler (remember there are three equations, for *i* = 1-3.

$$\delta u_i = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{e_{ij}}\delta x_j + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\omega_{ij}}\delta x_j$$

Let's call the first term e_{ij} and the 2nd term ω_{ij} to get

$$\delta u_i = (e_{ij} + \omega_{ij})\delta x_j$$

The e_{ij} term is the *strain tensor*. We'll write out a few terms

<u>*i*=1, *j*=1</u>

$$e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$$

<u>*i*=1, *j*=2</u>

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

You probably get the idea…. the strain tensor in all its glory looks like this:

$$e_{ij} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

In terms of *x*, *y*, *z* axes, we can rewrite the strain tensor to be

$$e_{ij} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Note that, just like the stress tensor, the strain tensor is symmetric. So

$$e_{ij} = e_{ji}$$

This means there are only six independent components. Also, like the stress tensor, the strain tensor can be rotated into a coordinate system of principle strains. The strain tensor also has similar invariants to the stress tensor (you shouldn't be surprised at this). The first invariant, the trace of the tensor, is this:

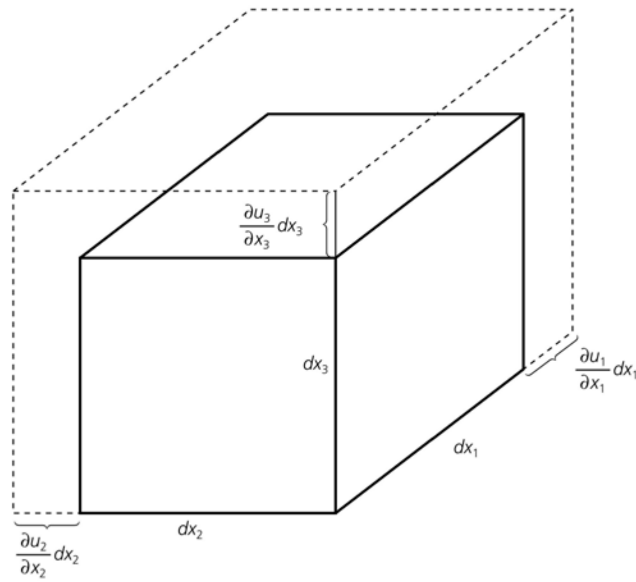
$$\theta = e_{ii} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \bullet \vec{u}$$

This quantity is known as the dilatation (i.e. the volumetric change associated with the deformation). The right hand side of the equation notes that the dilatation is equal to the divergence of the displacement vector field. Divergence (which is essentially defined above) measures sources and sinks in a vector field. A complimentary vector calculus operator is the curl $\vec{\nabla} \times \vec{u}$, which measures the rotation in a vector field. It turns out, that that any vector field can be decomposed entirely into curl and divergence terms.

Anyway, let's prove that the trace of the strain tensor is the relative volumetric change. It's actually pretty easy.

In the principle strain coordinate system, where there are no shear strains, the volumetric change is represented by the figure below

Figure 2.3-13: Change in volume due to principle strains.



So the volume of this cube after deformation is:

$$V + \Delta V = \left(1 + \frac{\partial u_1}{\partial x_1}\right) dx_1 \left(1 + \frac{\partial u_2}{\partial x_2}\right) dx_2 \left(1 + \frac{\partial u_3}{\partial x_3}\right) dx_3$$

Multiply this out to get

$$V + \Delta V = \left(1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right) dx_1 dx_2 dx_3$$

Which to first order is

$$V + \Delta V = \left(1 + \underbrace{\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}}_{\theta}\right) \underbrace{dx_1 dx_2 dx_3}_{V}$$

This is just

$$V + \Delta V = (1 + \theta) V$$

$$\theta = \frac{\Delta V}{V}$$

Hence, to first order, dilatation is just the relative change in volume. This points out a general trait of strain: it describes the relative change in length, not the absolute change in length. This means strain will have units length/length = no units.

1.1 Rotation

We've been talking about the strain tensor, e_{ij}; but there was another term that describes deformation: ω_{ij}.

By plugging numbers into the definition for ω_{ij}

$$\omega_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}$$

it can be seen that ω_{ij} has no terms on the diagonal, and the matrix is antisymmetric, which means $\omega_{ij} = -\omega_{ji}$. This implies there are only three independent terms. These three independent terms can be put into a vector $\vec{\omega}$, and the ω_{ij} tensor defines a rotation of magnitude $|\omega|$ in the direction of $\vec{\omega}$, where $\vec{\omega} = \vec{\Omega} = \omega_{23}\hat{x}_1 + \omega_{31}\hat{x}_2 + \omega_{12}\hat{x}_3 = \frac{1}{2}\vec{\nabla} \times \vec{u}$

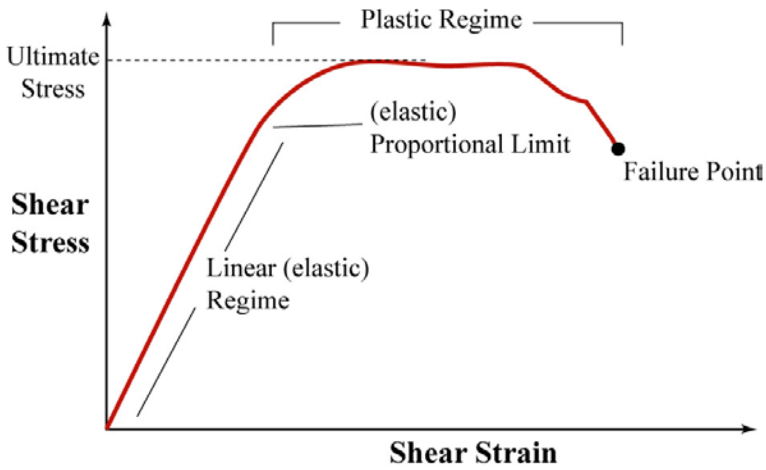
So, ***e*** defines the strain, and ***ω*** the rotation, of a body subjected to a stress, ***σ***. These are all defined in terms of the displacement field, **u**, which measures how different parts of a substance move in response to stress.

In the context of seismology, it will turn out that a change in volume, the divergence of the displacement field, propagates through the Earth at the P-wave velocity, and a rotation, the curl of the displacement field, propagates at the S-wave velocity.

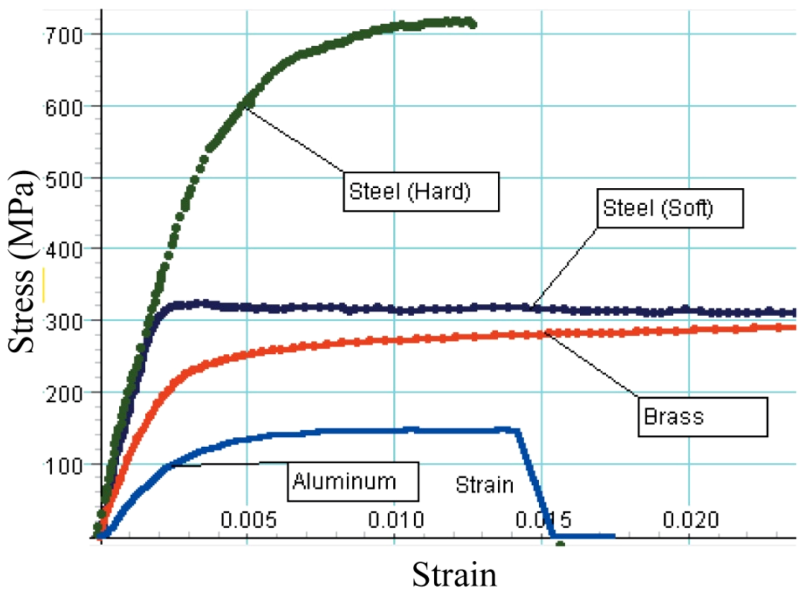
<u>A digression on deformation</u>

So, when a body is subjected to stress, the body undergoes deformations called strains. But, how can we calculate the strains when the stress is known? In any given material there is a relationship that relates stress and strain, which may be affected by several parameters, such as pressure, temperature and strain history. Nearly all earth materials have a ductile flow if small, steady stresses are applied for long lengths of time (e.g., diffusion creep). The same materials will fail brittlely or plastically if high stresses are applied.

In seismology and in parts of geodynamics, we are concerned with small-magnitude short duration stresses. For small stresses, we see observationally that a linear elastic relationship exists between stress and strain.



At high strains, deformation behaves plastically, until some eventual brittle failure. This of course depends on material:

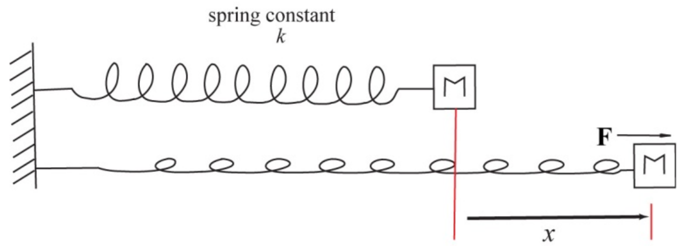


The implication is, for a purely elastic medium, reducing the applied stress in the linear elastic regime results in the medium restoring to its original shape. Also note: for perfect elasticity there is no energy loss as the material deforms in response to the applied stress.

<u>A digression on reference frames</u>
We have been discussing infinitesimal strain. When discussing larger strains, one typically has to decide whether the deformation is in respect to the the pre-strained locations. In this case we are using a *Lagrangian* reference frame. An alternative approach, is to view things in terms of their current locations, the *Eulerian* reference frame.

2.0 Constitutive Equations

This linear elastic relationship between stress and strain is described by a version of Hooke's law. You'll recall from freshman physics the 1-D Hooke's Law applied to, for example a spring with a spring constant *k*, and a mass M:



The relationship between an applied force **F** and displacement *x* due to this force is:

$$\mathbf{F} = -k\,x\ .$$

This is Hooke's law as you've learned it. We will be doing the same thing except for us:

$\mathbf{F} \Rightarrow$ stress ($\frac{\mathrm{F}}{Area}$ applied on some plane)

$k \Rightarrow$ c_{ijkl} (the 4th order elastic tensor, which describes elasticity of the medium)

$x \Rightarrow$ strain (deformation, instead of displacement, due to an applied stress)

Thus, the Hooke's law of elastic media – i.e., the relationship between the stress and strain tensors – is:

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

So, let's look further at the elastic tensor. First, recall repeated index notation. Again, a repeated index in a product indicates the sum is to be taken as the index varies from 1 to 3, e.g., (recall from last time, the dot product)

$$x_i \cdot y_i = \sum_{i=1}^{3} x_i \cdot y_i$$

$$x_1 y_1 + x_2 y_2 + x_3 y_3$$

Applying this to our Hooke's law, we have:

$$\begin{aligned} \sigma_{ij} &= c_{ijkl} e_{kl} \\ &= \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl} e_{kl} \end{aligned}$$

and so on. Let's look at just one term, say σ_{11}:

$$\begin{aligned}\sigma_{11} = c_{1111}e_{11} + c_{1112}e_{12} + c_{1113}e_{13} + \\ c_{1121}e_{21} + c_{1122}e_{22} + c_{1123}e_{23} + \\ c_{1131}e_{31} + c_{1132}e_{32} + c_{1133}e_{33}\end{aligned}$$

We have 8 more such equations for:

$$\left.\begin{matrix}\sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33}\end{matrix}\right\} \text{ each with 9 } c_{ijkl} \text{ terms.}$$

Adding these up, we have 9 equations, each with 9 c_{ijk1} terms, thus 9 x 9 = 81 terms.

2.1 Symmetry

But, recall our previous discussions about our medium not rotating, which gave:

$$\sigma_{ij} = \sigma_{ji} \Rightarrow 6 \text{ independent } \sigma \text{ terms}$$
$$e_{kl} = e_{lk} \Rightarrow 6 \text{ independent e terms.}$$

Because of this, the elastic tensor c_{ijkl} can be reduced to 6 x 6 = 36 independent terms, since

$$c_{ijkl} = c_{jikl}$$

and

$$c_{ijkl} = c_{ijlk} \, .$$

There also exists another symmetry relation from a thermodynamic consideration of a strain energy density function, which results in:

$$c_{ijkl} = c_{klij} \, .$$

This leaves us with only 21 independent terms, and this is the most general form of the elasticity tensor for elastic media.

2.2 Isotropy

If we assume that our medium is isotropic (i.e., the elastic properties of our medium are independent of direction or orientation of our material) then we can show that the elastic tensor has only *two* independent parameters, i.e., elastic moduli. This is the greatest reduction in the number of elastic constants. For sure, at very large scales, the Earth's interior is, to first order, isotropic, although there are regions within the Earth that are not. The c_{ijkl}'s for isotropic material can be defined in a number of ways, but a useful one is with *Lamé constants*. The Lamé constants are defined in terms of the c_{ijkl}:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Thus we can rewrite Hooke's law for an isotropic elastic solid ($\sigma_{ij} = c_{ijkl}e_{kl}$) as:

$$\begin{aligned}\sigma_{ij} &= \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \\ &= \lambda\theta\delta_{ij} + 2\mu e_{ij}\end{aligned}$$

where θ is the *dilatation:*

$$\begin{aligned}\theta &= e_{ii} \\ &\equiv Tr(e_{ij})\end{aligned}$$

$$\begin{aligned}&= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\ &= \vec{\nabla} \cdot \mathrm{u}\end{aligned}$$

We can spell out the equation at the top of the page, for an elastic isotropic solid as:

$$\begin{aligned}\sigma_{11} &= \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{11} \\ \sigma_{22} &= \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{22} \\ \sigma_{33} &= \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{33} \\ \sigma_{12} &= 2\mu e_{12} = \sigma_{21} \\ \sigma_{13} &= 2\mu e_{13} = \sigma_{31} \\ \sigma_{23} &= 2\mu e_{23} = \sigma_{32}\end{aligned}$$

λ does not have a straight-forward physical interpretation. μ, on the other hand, is the *shear modulus* or *rigidity*, and represents a material's shear strength. Another very important elastic constant is the *incompressibility*, or *bulk modulus*, K, which is related to the volume change due to pressure or compression. It can be defined in terms of λ and μ:

$$K = \lambda + \frac{2}{3}\mu \ .$$

K is the inverse of the volume change with pressure. So, something squishy that gets compressed readily with increasing pressure will have a low K. More quantitatively:

$$K = \frac{1}{-\frac{dP}{d\theta}}$$

Two other important parameters are *Young's modulus*, the ratio of tensional stress to the resulting extensional strain, and *Poisson's ratio*, which gives the ratio of the contraction along two axes to the extension along the third axis where tension is applied. Here is a series of relationships between most well-known moduli, from Stein and Wysession's seismology book, **Box 2.3-1** (pg. 51):

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{(3K-\lambda)} = \frac{E}{2\mu} - 1 = \frac{3K-2\mu}{2(3K+\mu)} = \frac{3K-E}{6K}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K(K-\lambda)}{3K-\lambda} = 2\mu(1+\nu) = \frac{9K\mu}{3K+\mu} = 3K(1-2\nu)$$

$$K = \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\nu)}{3\nu} = \frac{2\mu(1+\nu)}{3(1-2\nu)} = \frac{\mu E}{3(3\mu-E)} = \frac{E}{3(1-2\nu)}$$

$$\lambda = \frac{2\mu\nu}{1-2\nu} = \frac{\mu(E-2\mu)}{3\mu-E} = K - \frac{2}{3}\mu = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{3K\nu}{1+\nu} = \frac{3K(3K-E)}{9K-E}$$

$$\mu = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda) = \frac{E}{2(1+\nu)} = \frac{3K(1-2\nu)}{2(1+\nu)} = \frac{3KE}{9K-E}$$